Automated Reasoning Assignment – Stepan Sindelar

1. To show that $\neg p$ is logical consequence of the given set of formulas, we show that $\Phi \models \neg p$ where Φ is a concatenation of given formulas using $\wedge$, which is equivalent to show that $\Phi \wedge \neg\neg p$ is unsatisfiable.

   This can be done using resolution method, but first we need to convert the formula into a clause-set. We use classical transformation, namely

   1. shifting negation inwards, e.g. $\neg(q \wedge r)$ to $(\neg q \vee \neg r)$,
   2. convert $p \to \varphi$ to $\neg p \vee \varphi$,
   3. shifting disjunction inwards using distribution law,

   For the first formula, we have

$$p \to ((q \vee r) \wedge \neg(q \wedge r)) \overset{(1)}{\equiv} p \to ((q \vee r) \wedge (\neg q \vee \neg r)) \overset{(2)}{\equiv}$$
$$\overset{(2)}{\equiv} \neg p \vee ((q \vee r) \wedge (\neg q \vee \neg r)) \overset{(3)}{\equiv} (\neg p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$$

   The second one has exactly the same structure, so we have

$$p \to ((s \vee t) \wedge \neg(s \wedge t)) \equiv (\neg p \vee s \vee t) \wedge (\neg p \vee \neg s \vee \neg t)$$

   The remaining formulas are just implication, so using the second rule, we have

$$s \to q \equiv \neg s \vee q$$
$$\neg r \to t \equiv r \vee t$$
$$t \to s \equiv \neg t \vee s$$

   The final clause-set looks like

$$F = \{\{\neg p, q, r\}, \{\neg p, \neg q, \neg r\}, \{\neg p, s, t\}, \{\neg p, \neg s, \neg t\}, \{\neg s, q\}, \{r, t\}, \{\neg t, s\}, \{p\}\}$$

   Now we can apply resolution and try to derive empty clause, which means that the clause-set is unsatisfiable.

   $C_1 = \{p\}$ (axiom)
   $C_2 = \{\neg p, q, r\}$ (axiom)
   $C_3 = \{q, r\}$ (from C_1 and C_2)
   $C_4 = \{\neg p, \neg q, \neg r\}$ (axiom)
   $C_5 = \{\neg q, \neg r\}$ (from C_1 and C_4)

   $C_6 = \{\neg p, s, t\}$ (axiom)
   $C_7 = \{s, t\}$ (from C_1 and C_6)
   $C_8 = \{\neg p, \neg s, \neg t\}$ (axiom)
   $C_9 = \{\neg s, \neg t\}$ (from C_1 and C_8)

   $C_{10} = \{\neg t, s\}$ (axiom)

   $C_{11} = \{s\}$ (from C_7 and C_{10})
   $C_{12} = \{\neg t\}$ (from C_9 and C_{10})

   $C_{13} = \{\neg s, q\}$ (axiom)
   $C_{14} = \{q\}$ (from C_{11} and C_{13})
   $C_{15} = \{\neg r\}$ (from C_{14} and C_5)

   $C_{16} = \{r, t\}$ (axiom)
   $C_{17} = \{t\}$ (from C_{15} and C_{16})
   $C_{18} = \square$ (from C_{17} and C_{12})

   hence $F \vdash_R \square$

2. To show that F is renameable Horn, we construct clause-set $R(F)$ and then we show that $R(F)$ is satisfiable. $R(F)$ is defined as follows

$$R(F) = \{\{l, m\} \mid \quad l \text{ and } m \text{ are in the same clause } C \in F\}$$

   which in our case yields

$$\{\{\neg x, y\}, \{y, \neg u\}, \{\neg x, \neg u\}, \{\neg t, z\}, \{\neg y, z\}, \{z, t\}, \{\neg y, t\}, \{x, y\}, \{y, z\}, \{x, z\}, \{\neg z, \neg u\}\}$$

   Because $R(X)$ is an instance of 2-SAT for any X, we can use the algorithm from lecture: construct implications graph and see if we can find any directed path from x to $\neg x$ and $\neg x$ to x for any variable x. The implication graph is depicted in figure 1.

   Now we have to inspect every pair of variables:

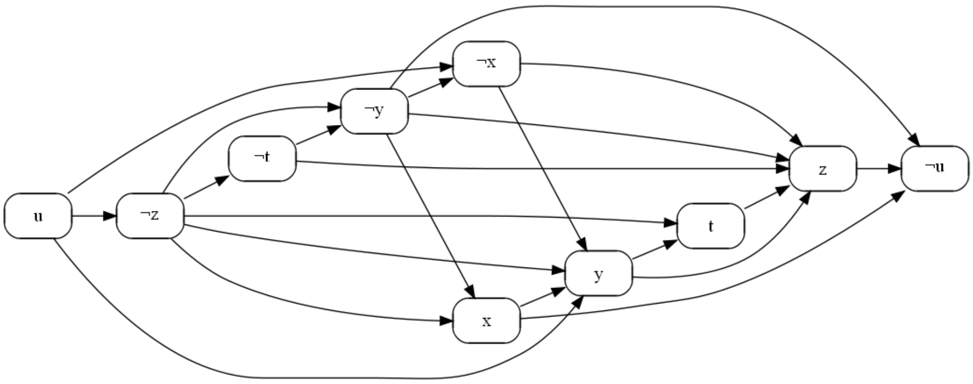

Figure 1: The implication graph for $R(F)$.

- u and $\neg u$: u does not have any incoming edges, therefore there cannot be any path from $\neg u$ to u.
- z and $\neg z$: the only incoming edge to $\neg z$ is from u and, following the same argument, there cannot be any path from z to $\neg z$.
- t and $\neg t$: the only incoming edge to $\neg t$ is from $\neg z$ and, again following the same argument, there cannot be path from t to $\neg t$.
- x and $\neg x$: all the directed paths from x end up in $\neg u$, which has no outcoming edges (and any of these paths does not visit $\neg x$).
- y and $\neg y$: exactly the same argument as for x and $\neg x$.

Renameable Horn clause-set F is unsatisfiable if and only if $F \vdash_{UR} \square$. Each step of the unit resolution is on a separate line.

$$\begin{aligned}
\{\{\neg x, y, \neg u\}, \{\neg t, z\}, \{\neg y, z, t\}, \{u\}, \{x, y, z\}, \{\neg z, \neg u\}\} \\
\{\{\neg x, y\}, \{\neg t, z\}, \{\neg y, z, t\}, \{x, y, z\}, \{\neg z\}\} \\
\{\{\neg x, y\}, \{\neg t\}, \{\neg y, t\}, \{x, y\}\} \\
\{\{\neg x, y\}, \{\neg y\}, \{x, y\}\} \\
\{\{\neg x\}, \{x\}\} \\
\square
\end{aligned}$$

Construction of an implication graph is linear. To check whether there are directed paths between a pair of complementary literals is linear w.r.t. the size of the graph and because we have to do it for all possible pairs of complementary literals, in total it is polynomial. Unit resolution is polynomial, because in each step we eliminate at least one clause, and in each step we have to go through the whole clause-set.

3. Example clause-set

$$F = \{\{x\}, \{\neg x, y\}, \{x, \neg y\}\}$$

From the equalities below

$$\begin{aligned}
PL(F) =& F \quad \text{(there is no pure literal, both } x \text{ and } y \text{ appear also as negated)} \\
UP(F) =& \{\{y\}\} \quad \text{(unit clause } \{x\}\text{)} \\
PL(UP(F)) =& \{\} \quad \text{(now } y \text{ was pure literal)}
\end{aligned}$$

It follows that

$$\begin{aligned}
PL(UP(PL(F))) =& \{\} \\
UP(PL(F)) =& \{\{y\}\} \\
PL(UP(PL(F))) \neq& UP(PL(F))
\end{aligned}$$

4. Firstly, we change the implication to disjunction

$$\exists x \forall y \exists z \left[R(x, y) \rightarrow \forall x \exists y \left(S(x, y) \wedge R(y, f(z))\right)\right] \equiv \exists x \forall y \exists z \left[\neg R(x, y) \vee \forall x \exists y \left(S(x, y) \wedge R(y, f(z))\right)\right]$$

Then we rename the variables in the second part of the disjunction so that x and y do not conflict with the outer x and y. z is free in this sub-formula, so we shouldn't rename it.

$$\exists x \forall y \exists z \left[\neg R\left(x,y\right) \vee \forall r \exists t \left(S(r,t) \wedge R\left(t, f\left(z\right)\right)\right)\right]$$

Now, we can move the quantifiers outwards

$$\exists x \forall y \exists z \left[\neg R\left(x,y\right) \vee \forall r \exists t \left(S(r,t) \wedge R\left(t, f\left(z\right)\right)\right)\right] \equiv \exists x \forall y \exists z \forall r \exists t \left[\neg R\left(x,y\right) \vee \left(S(r,t) \wedge R\left(t, f\left(z\right)\right)\right)\right]$$

To obtain the Skolem standard form, we introduce a skolem constant c_x to eliminate existentially quantified x, unary skolem functions f_z to eliminate z and binary skolem function f_t to eliminate t.

$$\forall y \forall r \left[\neg R\left(c_x, y\right) \vee \left(S(r, f_t(y,r)) \wedge R\left(f_t(y,r), f\left(f_z(y)\right)\right)\right)\right]$$

To obtain a clausal normal form, we need to transform matrix of this formula to CNF. We use the rule (3) from the first exercise.

$$\left[\neg R\left(c_x, y\right) \vee \left(S(r, f_t(y,r)) \wedge R\left(f_t(y,r), f\left(f_z(y)\right)\right)\right)\right] \equiv$$
$$\left[\neg R\left(c_x, y\right) \vee S(r, f_t(y,r))\right] \wedge \left[\neg R\left(c_x, y\right) \vee \left(R\left(f_t(y,r), f\left(f_z(y)\right)\right)\right)\right]$$

Finally, the clausal normal form looks like

$$\left\{\left\{\neg R\left(c_x, y\right), S(r, f_t(y,r))\right\}, \left\{\neg R\left(c_x, y\right), R\left(f_t(y,r), f\left(f_z(y)\right)\right)\right\}\right\}$$

5. First, we need to construct H-universe of F. Because we have only one constant and only one unary function, we get $H_0 = \{a\}$ and in general $H_{i+1} = H_i \cup \left\{f^{i+1}(a)\right\}$ where $f^k(a)$ denotes k chained applications of function f to a, e.g. $f^2(a) = f(f(a))$.

   The atom set of F is

$$\{T(a), S(a,a), T(f(a)), S(a, f(a)), S(f(a), a), S(f(a), f(a)), T(f(f(a))), \ \ldots\}$$

   Our closed semantic tree is depicted in figure 2. Every failure vertex is connected to a box, which contains a ground instance that is falsified by the set of the labels on the path from the root to this vertex.

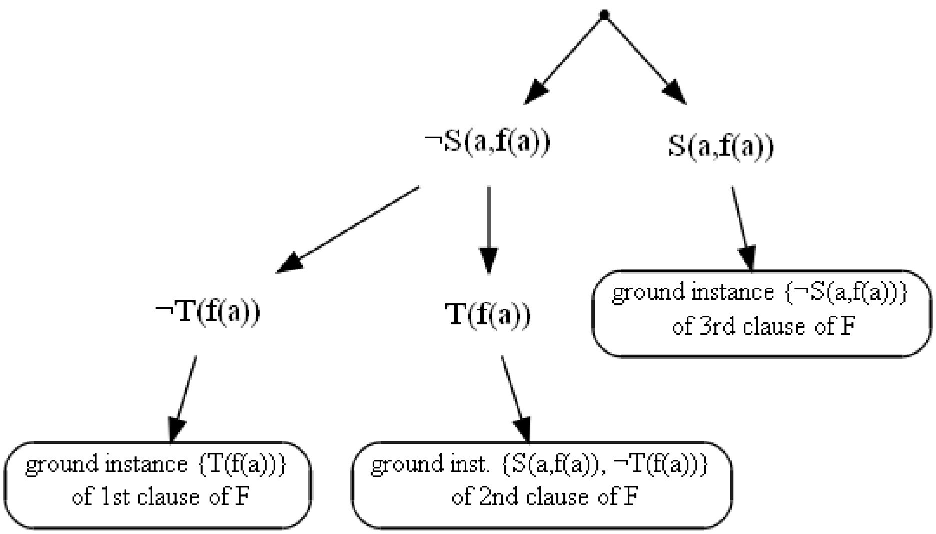

Figure 2: A closed semantic tree for F.